CHAIN CONDITIONS IN DEPENDENT GROUPS

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Abstract. In this note we prove and disprove some chain conditions in type definable and definable groups in dependent, strongly dependent and strongly^2 dependent theories.

1. Introduction

This note is about chain conditions in dependent, strongly dependent and strongly^2 dependent theories.

Throughout, all formulas will be first order, T will denote a complete first order theory, and $\mathfrak{C}$ will be the monster model of T — a very big saturated model that contains all small models. We do not differentiate between finite tuples and singletons unless we state it explicitly.

Definition 1.1. A formula $\varphi(x, y)$ has the independence property in some model if for every $n < \omega$ there are $\langle a_i, b_s \,| i < n, s \subseteq n \rangle$ such that $\varphi(a_i, b_s)$ holds iff $i \in s$.

A (first order) theory T is dependent (sometimes also NIP) if it does not have the independence property: there is no formula $\varphi(x, y)$ that has the independence property in any model of T. A model M is dependent if $Th(M)$ is.

For a good introduction to dependent theories appears we recommend [Adl08], but we shall give an exact reference to any fact we use, so no prior knowledge is assumed.

What do we mean by a chain condition? rather than giving an exact definition, we give an example of such a condition — the first one. It is the Baldwin-Saxl Lemma, which we shall present with the (very easy and short) proof.

Definition 1.2. Suppose $\varphi(x, y)$ is a formula. Then if G is a definable group in some model, and for all $c \in C$, $\varphi(x, c)$ defines a subgroup, then $\{\varphi(\mathfrak{C}, c) \,|\, c \in C\}$ is a family of *uniformly definable subgroups.*

Lemma 1.3. [BS76] *Let G be a group definable in a dependent theory. Suppose $\varphi(x, y)$ is a formula and that $\{\varphi(x, c) \,|\, c \in C\}$ defines a family of subgroups of G. Then there is a number $n < \omega$ such that any finite intersection of groups from this family is already an intersection of n of them.*

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Proof. Suppose not, then for every $n < \omega$ there are $c_0, \ldots, c_{n-1} \in C$ and $g_0, \ldots, g_{n-1} \in G$ (in some model) such that $\varphi(g_i, c_j)$ holds iff $i \neq j$. For $s \subseteq n$, let $g_s = \prod_{i \in s} g_i$ (the order does not matter), then $\varphi(g_s, c_j)$ iff $j \notin s$ — this is a contradiction. □

In stable theories (which we shall not define here), the Baldwin-Saxl lemma is even stronger: every intersection of such a family is really a finite one (see [Poi01, Proposition 1.4]).

The focus of this note is type definable groups in dependent theories, where such a proof does not work.

Definition 1.4. A *type definable group* for a theory T is a type — a collection $\Sigma(x)$ of formulas (maybe over parameters), and a formula $\nu(x, y, z)$, such that in the monster model $\mathfrak{C}$ of T, $\langle \Sigma(\mathfrak{C}), \nu \rangle$ is a group with ν defining the group operation (without loss of generality, $T \models \forall xy \exists^{\leq 1} z (\nu(x, y, z))$). We shall denote this operation by $\cdot$.

In stable theories, their analysis becomes easier as each type definable group is an intersection of definable ones (see [Poi01]).

Remark 1.5. In this note we assume that G is a finitary type definable group, i.e. x above is a finite tuple.

Definition 1.6. Suppose $G \geq H$ are two type definable groups (H is a subgroup of G). We say that the index $[G : H]$ is *unbounded*, or ∞, if for any cardinality κ, there exists a model $M \models T$, such that $\left[G^M : H^M\right] \geq \kappa$. Equivalently (by the Erdős-Rado coloring theorem), this means that there exists (in $\mathfrak{C}$) a sequence of indiscernibles $\langle a_i \mid i < \omega \rangle$ (over the parameters defining G and H) such that $a_i \in G$ for all i, and $i < j \Rightarrow a_i \cdot a_j^{-1} \notin H$. In $\mathfrak{C}$, this means that $\left[G^{\mathfrak{C}} : H^{\mathfrak{C}}\right] = |\mathfrak{C}|$. When G and H are definable, then by compactness this is equivalent to the index $[G : H]$ being infinite.

So $[G : H]$ is *bounded* if it is not unbounded.

This leads to the following definition

Definition 1.7. Let G be a type definable group.

(1) For a set A, G_A^{00} is the minimal A-type definable subgroup of G of bounded index.
(2) We say that G^{00} exists if $G_A^{00} = G_\emptyset^{00}$ for all A.

Shelah proved

Theorem 1.8. [She08] If G is a type definable group in a dependent theory, then G^{00} exists.

Even though fields are not the main concern of this note, the following question is in the basis of its motivation. Recall

Theorem 1.9. [Lan02, Theorem VI.6.4] *(Artin-Schreier) Let* k *be a field of characteristic* p*. Let* ρ *be the polynomial* $X^p - X$*.*

(1) *Given* $a \in k$*, either the polynomial* $\rho - a$ *has a root in* k*, in which case all its root are in* k*, or it is irreducible. In the latter case, if* α *is a root then* $k(\alpha)$ *is cyclic of degree* p *over* k*.*
(2) *Conversely, let* K *be a cyclic extension of* k *of degree* p*. Then there exists* $\alpha \in K$ *such that* $K = k(\alpha)$ *and for some* $a \in k$*,* $\rho(\alpha) = a$*.*

Such extensions are called Artin-Schreier extensions.

The first author, in a joint paper with Thomas Scanlon and Frank Wagner, proved

Theorem 1.10. [KSW11] *Let* K *be an infinite dependent field of characteristic* $p > 0$*. Then* K *is Artin-Schreier closed — i.e.* ρ *is onto.*

What about the type definable case? What if K is an infinite type definable field?

In simple theories (which we shall not define), we have:

Theorem 1.11. [KSW11] *Let* K *be a type definable field in a simple theory. Then* K *has boundedly many AS extensions.*

But for the dependent case we only proved

Theorem 1.12. [KSW11] *For an infinite type definable field* K *in a dependent theory there are either unboundedly many Artin-Schreier extensions, or none.*

from these two we conclude

Corollary 1.13. *If* T *is stable (so it is both simple and dependent), then type definable fields are AS closed.*

The following, then, is still open

Question 1.14. *What about the dependent case? In other words, is it true that infinite type definable fields in dependent theories are AS-closed?*

Observing the proof of Theorem 1.10, we see that it is enough to find a number n, and $n+1$ algebraically independent elements, $\langle a_i \,|\, i \leq n \rangle$ in $k := K^{p^\infty}$, such that $\bigcap_{i<n} a_i \rho(K) = \bigcap_{i \leq n} a_i \rho(K)$. So the Baldwin-Saxl applies in the case where the field K is definable. If K is type definable, we may want something similar. But what can we prove?

A conjecture of Frank Wagner is the main motivation question

Conjecture 1.15. *Suppose* T *is dependent, then the following holds*

☺ *Suppose* G *is a type definable group. Suppose* $p(x, y)$ *is a type and* $\langle a_i \,|\, i < \omega \rangle$ *is an indiscernible sequence such that* $G_i = p(x, a_i) \leq G$*. Then there is some* n*, such that for all finite sets,* $v \subseteq \omega$*, the intersection* $\bigcap_{i \in v} G_i$ *is equal to a sub-intersection of size* n*.*

Let refer to ☺ as *Property A* (of a theory T) for the rest of the paper. So we have

Fact 1.16. *If Property A is true for a theory* T*, then type definable fields are Artin-Schreier closed.*

In Section 2, we deal with strongly^2 dependent theories (this is a much stronger condition than merely dependence), and among other things, prove that Property A is true for them.

In Section 3, we give some generalizations and variants of Baldwin-Saxl for type definable groups in dependent and strongly dependent theories (which we define below). One of them is joint work with Frank Wagner. We prove that Property A holds for theories with bounded dp-rank.

In Section 4, we provide a counterexample that shows that property A does not hold in stable theories, so Conjecture 1.15 as it is stated is false.

Question 1.17. *Does Property A hold for strongly dependent theories?*

2. Strongly^2 dependent theories

Notation 2.1. We call an array of elements (or tuples) $\langle a_{i,j} \,|\, i, j < \omega \rangle$ an *indiscernible array* over A if for $i_0 < \omega$, the i_0-row $\langle a_{i_0,j} \,|\, j < \omega \rangle$ is indiscernible over the rest of the sequence ($\{a_{i,j} \,|\, i \neq i_0, i, j < \omega\}$) and A, i.e. when the rows are mutually indiscernible.

Definition 2.2. A theory T is said to be not *strongly^2 dependent* if there exists a sequence of formulas $\langle \varphi_i(x, y_i, z_i) \,|\, i < \omega \rangle$, an array $\langle a_{i,j} \,|\, i, j < \omega \rangle$ and $b_k \in \{a_{i,j} \,|\, i < k, j < \omega\}$ such that

- The array $\langle a_{i,j} \,|\, i, j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x, a_{i,0}, b_i) \wedge \neg\varphi_i(x, a_{i,1}, b_i) \,|\, i < \omega\}$ is consistent.

So T is *strongly^2 dependent* when this configuration does not exist.

Note that the roles of i and j are not symmetric.

(In the definition above, x, z_i, y_i can be tuples, the length of z_i and y_i may depend on i).

This definition was introduced and discussed in [Shec] and [Shea].

Remark 2.3. By [Shec, Claim 2.8], we may assume in the definition above that x is a singleton.

Proposition 2.4. *Suppose* T *is strongly^2 dependent, then it is impossible to have a sequence of type definable groups* $\langle G_i \,|\, i < \omega \rangle$ *such that* $G_{i+1} \leq G_i$ *and* $[G_i : G_{i+1}] = \infty$.

Proof. Without loss of generality, we shall assume that all groups are definable over $\emptyset$. Suppose there is such a sequence $\langle G_i \,|\, i < \omega \rangle$. Let $\langle a_{i,j} \,|\, i, j < \omega \rangle$ be an indiscernible array such that for each $i < \omega$, the sequence $\langle a_{i,j} \,|\, j < \omega \rangle$ is a sequence from G_i (in $\mathfrak{C}$) such that $a_{i,j'}^{-1} \cdot a_{i,j} \notin G_{i+1}$

for all $j < j' < \omega$. We can find such an array because of our assumption and Ramsey (for more detail, see the proof of Corollary 2.8 below).

For each $i < \omega$, let $\psi_i(x)$ be in the type defining G_{i+1} such that $\neg\psi_i\left(a_{i,j'}^{-1} \cdot a_{i,j}\right)$. By compactness, there is a formula $\xi_i(x)$ in the type defining G_{i+1} such that for all $a, b \in \mathfrak{C}$, if $\xi_i(a) \wedge \xi_i(b)$ then $\psi_i\left(a \cdot b^{-1}\right)$ holds. Let $\varphi_i(x,y,z) = \xi_i\left(y^{-1} \cdot z^{-1} \cdot x\right)$. For $i < \omega$, let $b_i = a_{0,0} \cdot \ldots \cdot a_{i-1,0}$ (so $b_0 = 1$).

Let us check that the set $\{\varphi_i(x, a_{i,0}, b_i) \wedge \neg\varphi_i(x, a_{i,1}, b_i) \mid i < \omega\}$ is consistent. Let $i_0 < \omega$, and let $c = b_{i_0}$. Then for $i < i_0$, $\varphi_i(c, a_{i,0}, b_i)$ holds iff $\xi_i(a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0})$ but the product $a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0}$ is an element of G_{i+1} and ξ_i is in the type defining G_{i+1}, so $\varphi_i(c, a_{i,0}, b_i)$ holds. Now, $\varphi_i(c, a_{i,1}, b_i)$ holds iff $\xi_i\left(a_{i,1}^{-1} a_{i,0} \cdot \ldots \cdot a_{i_0-1,0}\right)$. However, since $\xi_i(a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0})$ holds, by choice of ξ_i we have

$$\psi_i\left(\left[a_{i,1}^{-1} a_{i,0} \cdot \ldots \cdot a_{i_0-1,0}\right] \cdot \left[a_{i+1,0} \cdot \ldots \cdot a_{i_0-1,0}\right]^{-1}\right)$$

i.e. $\psi_i\left(a_{i,1}^{-1} \cdot a_{i,0}\right)$ holds — contradiction. □

Remark 2.5. It is well known (see [Poi01]) that in superstable theories the same proposition hold.

The next corollary already appeared in [Shec, Claim 0.1] with definable groups instead of type definable (with proof already in [Shea, Claim 3.10]).

Corollary 2.6. *Assume T is strongly[2] dependent. If G is a type definable group and h is a definable homomorphism $h : G \to G$ with finite kernel then h is almost onto G, i.e., the index $[G : h(G)]$ is bounded (i.e. $< \infty$). If G is definable, then the index must be finite.*

Proof. Consider the sequence of groups $\left\langle h^{(i)}(G) \mid i < \omega \right\rangle$ (i.e. G, $h(G)$, $h(h(G))$, etc.). By Proposition 2.4, for some $i < \omega$, $\left[h^{(i)}(G) : h^{(i+1)}(G)\right] < \infty$. Now the Corollary easily follows from

Claim. If G is a group, $h : G \to G$ a homomorphism with finite kernel, then $[G : h(G)] + \aleph_0 = [h(G) : h(h(G))] + \aleph_0$.

Proof. (of claim) Let $H = h(G)$. Easily, one has $[H : h(H)] \leq [G : H]$.

We may assume that $[G : H]$ is infinite. Let $\ker(h) = \{g_0, \ldots, g_{k-1}\}$. Suppose that $[G : H] = \kappa$ but $[H : h(H)] < \kappa$. So let $\{a_i \mid i < \kappa\} \subseteq G$ are such that $a_i^{-1} \cdot a_j \notin h(G)$ for $i \neq j$. So there must be some coset $a \cdot h(H)$ in H such that for infinitely many $i < \kappa$, $h(a_i) \in a \cdot h(H)$. Let us enumerate them as $\langle a_i \mid i < \omega\rangle$. So for $i < j < \omega$, let $C(a_i, a_j)$ be the least number $l < k$ such that there is some $y \in h(G)$ with $y^{-1} a_i^{-1} a_j = g_l$. By Ramsey, we may assume that $C(a_i, a_j)$ is constant. Now pick $i_1 < i_2 < j < \omega$. So we have $y^{-1} a_{i_1}^{-1} a_j = (y')^{-1} a_{i_2}^{-1} a_j$, so $y^{-1} a_{i_1}^{-1} = (y')^{-1} a_{i_2}^{-1}$ and hence $a_{i_1}^{-1} a_{i_2} = y(y')^{-1} \in h(G)$ — contradiction.

□

Corollary 2.7. *If* K *is a strongly*^{2} *dependent field, (or even a type definable field in a strongly*^{2} *dependent theory) then for all* $n<\omega$, $\left[K^\times:(K^\times)^n\right]<\infty$.

Corollary 2.8. *Let* G *be type definable group in a strongly*^{2} *dependent theory* T.

(1) Given a family of uniformly type definable subgroups $\{p(x,a_i)\,|\,i<\omega\}$ such that $\langle a_i\,|\,i<\omega\rangle$ is an indiscernible sequence, there is some $n<\omega$ such that $\bigcap_{j<\omega}p(\mathfrak{C},a_j)=\bigcap_{j<n}p(\mathfrak{C},a_j)$. In particular, T has Property A.

(2) Given a family of uniformly definable subgroups $\{\varphi(x,c)\,|\,c\in C\}$, the intersection

$$\bigcap_{c\in C}\varphi(\mathfrak{C},c)$$

is already a finite one.

Proof. (1) Assume without loss of generality that G is defined over $\emptyset$. Let $G_i=p(\mathfrak{C},a_i)$, and let $H_i=\bigcap_{j<i}G_i$. By Proposition 2.4, for some $i_0<\omega$, $[H_{i_0}:H_{i_0+1}]<\infty$. For $r\geq i_0$, let $H_{i_0,r}=\bigcap_{j<i_0}G_j\cap G_r$ (so $H_{i_0+1}=H_{i_0,i_0}$). By indiscerniblity, $[H_{i_0}:H_{i_0,r}]<\infty$. This means (by definition of $H_{i_0}^{00}$) that $H_{i_0}^{00}\leq H_{i_0,r}$ for all $r>i_0$. However, if $H_{i_0,i_0}\neq H_{i_0,r}$ for some $i_0<r<\omega$, then by indiscerniblity $H_{i_0,r}\neq H_{i_0,r'}$ for all $i_0\leq r<r'$, and by compactness and indiscerniblity we may increase the length ω of the sequence to any cardinality κ, so that the size of $H_{i_0}/H_{i_0}^{00}$ is unbounded — contradiction. This means that $H_{i_0+1}\subseteq G_r$ for all $r>i_0$, and so $\bigcap_{i<\omega}G_i=\bigcap_{i<i_0+1}G_i$.

(2) Assume not. Then we can find a sequence $\langle c_i\,|\,i<\omega\rangle$ of element of C such that $\bigcap_{j<i}\varphi(\mathfrak{C},c_j)\neq\bigcap_{j<i+1}\varphi(\mathfrak{C},c_j)$. By Ramsey, we can extract an indiscernible sequence $\langle a_i\,|\,i<\omega\rangle$ such that for any n, and any formula $\psi(x_0,\ldots,x_{n-1})$, if $\psi(a_0,\ldots,a_{n-1})$ holds then there are $i_0<\ldots<i_{n-1}$ such that $\psi(c_{i_0},\ldots,c_{i_{n-1}})$ holds. In particular, $\varphi(\mathfrak{C},a_i)$ defines a subgroup of G and $\bigcap_{j<i}\varphi(\mathfrak{C},a_j)\neq\bigcap_{j<i+1}\varphi(\mathfrak{C},a_j)$. But this contradicts (1). □

As further applications, we show that some theories are not strongly^2 dependent.

Example 2.9. Suppose $\langle G,+,<\rangle$ is an ordered abelian group. Then its theory $Th(G,+,0,<)$ is not strongly^2 dependent.

Proof. We work in the monster model $\mathfrak{C}$. Let $G_d=\{x\in\mathfrak{C}\,|\,\forall n<\omega\,(n\,|\,x)\}$, so it is a divisible ordered subgroup of G. Note that since G is ordered, it is torsion free, so really it is a $\mathbb{Q}$-vector space. Define a descending sequence of infinite type definable groups $G_d^i\leq G_d$ for $i<\omega$ such that $\left[G_d^i:G_d^{i+1}\right]=\infty$. This contradicts Proposition 2.4. Let $G_d^0=G_d$, and suppose we have chosen G_d^i. Let $a_i\in G_d^i$ be positive. Let $G_d^{i+1}=G_d^i\cap\bigcap_{n<\omega}(-a_i/n,a_i/n)$. This is a type definable subgroup of G_d^i. The sequence $\langle k\cdot a_i\,|\,k<\omega\rangle$ satisfies $(k-l)\cdot a_i\notin(-a_i/2,a_i/2)$ for any $k\neq l$, and by Ramsey (as in the proof of Corollary 2.8 (2)) we get $\left[G_d^i:G_d^{i+1}\right]=\infty$. □

Example 2.10. The theory $\mathrm{Th}(\mathbb{R},+,\cdot,0,1)$ is strongly dependent (it is even o-minimal, so dp-minimal — see Definitions 3.7 and 3.5 below). However it is not strongly^2 dependent.

Example 2.11. The theory $\mathrm{Th}(\mathbb{Q}_p,+,\cdot,0,1)$ of the p-adics is strongly dependent (it is also dp-minimal), but not strongly^2 dependent: The valuation group $(\mathbb{Z},+,0,<)$ is interpretable.

Adding some structure to an algebraically closed field, we can easily get a strongly^2 dependent theory.

Example 2.12. Let $L = L_{\text{rings}} \cup \{P,<\}$ where L_{rings} is the language of rings $\{+,\cdot,0,1\}$, P is a unary predicate and $<$ is a binary relation symbol. Let K be $\mathbb{C}$ (so is an algebraically closed field), and let $P \subseteq K$ be a countable set of algebraically independent elements, enumerated as $\{a_i \mid i \in \mathbb{Q}\}$. Let $M = \langle K,P,<\rangle$ where $a <^M b$ iff $a,b \in P$ and $a = a_i$, $b = a_j$ where $i < j$. Let $T = \mathrm{Th}(M)$.

Claim 2.13. T is strongly^2 dependent.

Proof. Note that T is axiomatizable by saying that the universe is an algebraically closed field, P is a subset of algebraically independent elements and $<$ is a dense linear order on P (to see this, take two saturated models of the same size and show that they are isomorphic).

Let us fix some terminology:

- When we write acl, we mean the algebraic closure in the field sense. When we say basis, we mean a transcendental basis.
- When we say that a set is independent / dependent over A for some set A, we mean that it is dependent / independent in the pregeometry induced by $\mathrm{cl}(X) = \mathrm{acl}(AX)$.
- $\mathrm{dcl}(X)$ stands for the definable closure of X.

We work in a saturated model $\mathfrak{C}$ of T.

Suppose X is some set. Let X_0 be some basis for X over P, and let $\mathrm{dcl}^P(X)$ be the set of $p \in P$ such that there exists some minimal finite $P_0 \subseteq P$ with $p \in P_0$ and some $x \in X$ such that $x \in \mathrm{acl}(P_0X_0)$. Note that this set is contained in $\mathrm{dcl}(X)$ (since P is linearly ordered) and that it does not depend on the choice of X_0.

Suppose a is a finite tuple, and A is a set. Let $A^P = \mathrm{dcl}^P(A)$.

Let $\mathrm{tp}_K(a/A)$ the type of $a \frown (Aa)^P$ (considered as a tuple, ordered by $<^{\mathfrak{C}}$) over $A \cup A^P$ in the field language, and $\mathrm{tp}_P(a/A)$ the type of the tuple $(Aa)^P$ over A^P in the order language.

Subclaim. For finite tuples a,b and a set A, $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$ iff $\mathrm{tp}_P(Aa/A) = \mathrm{tp}_P(Ab/A)$ and $\mathrm{tp}_K(a/A) = \mathrm{tp}_K(b/A)$. In fact, in this case, there is an automorphism of the field $\mathrm{acl}(abAP)$ fixing A pointwise and P setwise taking a to b. This automorphism is an elementary map.

Proof. Given that the P and K types are equal, it is easy to construct an automorphism of $\mathrm{acl}(abAP)$ as above. First we construct an automorphism of $\langle P,<\rangle$ that takes a^P to b^P and

fixes A^P. We can extend this automorphism to A_0P where A_0 is a basis of A over P. By definition of dcl^P, we can also extend it to $\mathrm{acl}(AP)$, fixing A pointwise. Let $a' \subseteq a$ be a basis for a over AP, and $b' \subseteq b$ a basis for b over AP. By definition of dcl^P, $|a'| = |b'|$. This means we can extend this automorphism to $\mathrm{acl}(aAP)$, taking it to $\mathrm{acl}(bAP)$. Next extend this to an automorphism of $\mathrm{acl}(abAP)$ (possible since both a and b are finite). Now we can extend this to an automorphism of $\mathfrak{C}$ since it is algebraically closed. Note that if $c \notin \mathrm{acl}(abAP)$, we can choose this automorphism to fix c. □

Suppose that $\langle a_{i,j} \,|\, i,j < \omega \rangle$ is an indiscernible array over a parameter set A as in Definition 2.2 and that c is a singleton such that:

- The sequence $I_0 := \langle a_{0,j} \,|\, j < \omega \rangle$ is not indiscernible over c, and moreover $\mathrm{tp}(a_{0,0}/c) \neq \mathrm{tp}(a_{0,1}/c)$.
- For $i > 0$, the sequence $I_i := \langle a_{i,j} \,|\, j < \omega \rangle$ is not indiscernible over $c \cup \bigcup_{k<i} I_k \cup A$.

Suppose first that $c \notin \mathrm{acl}(APa_{0,0}a_{0,1})$. Then, by the proof of the first subclaim, we get a contradiction, since there is an automorphism fixing cA pointwise and P setwise taking $a_{0,0}$ to $a_{0,1}$. So $c \in \mathrm{acl}(APa_{0,0}a_{0,1})$. Increase the parameter set A by adding the first row $\langle a_{0,j} \,|\, j < \omega \rangle$. So we may assume that $c \in \mathrm{acl}(AP)$. Choose a basis $A_0 \subseteq A$ over P, and let $c^P \subseteq P$ the unique minimal tuple of elements such that $c \in \mathrm{acl}\left(A_0 c^P\right)$. Since $c \in \mathrm{acl}\left(Ac^P\right)$, we may replace c by c^P and assume that c is a tuple of elements in P (here we use the fact that if I is indiscernible over Ac^P then it is also indiscernible over $\mathrm{acl}\left(Ac^P\right)$).

Expand all the sequences to order type $\omega^* + \omega + \omega$. Let $B = \bigcup \{a_{i,j} \,|\, i < \omega, j < 0 \vee \omega \leq j\} \cup A$. For each $i < \omega$ and $0 \leq j < \omega$, let $a^P_{i,j}$ be $\mathrm{dcl}^P(a_{i,j}B)$ considered as a tuple ordered by $<^{\mathfrak{C}}$, and let $B^P = \mathrm{dcl}^P(B)$. Then $\left\langle a^P_{i,j} \,|\, i,j < \omega \right\rangle$ is an indiscernible array over B^P and $\left\langle a_{i,j} \frown a^P_{i,j} \,|\, i,j < \omega \right\rangle$ is an indiscernible array over $B \cup B^P$.

As both the theories of dense linear orders and algebraically closed fields are strongly[2] dependent (this is easy to check), there is some i_0 such that $\left\langle a^P_{i_0,j} \,|\, j < \omega \right\rangle$ is indiscernible over $cB^P \cup \left\{ a^P_{i,j} \,|\, i < i_0, j < \omega \right\}$ in the order language and $\left\langle a_{i_0,j} \frown a^P_{i_0,j} \,|\, j < \omega \right\rangle$ is indiscernible over $cB \cup B^P \cup \left\{ a_{i,j} \frown a^P_{i,j} \,|\, i < i_0, j < \omega \right\}$ in the field language.

Let $C = \bigcup \{a_{i,j} \,|\, i < i_0, j < \omega\}$. We must check that $\langle a_{i_0,j} \,|\, j < \omega \rangle$ is indiscernible over BCc. Let us show, for instance, that $\mathrm{tp}(a_{i_0,0}/BCc) = \mathrm{tp}(a_{i_0,1}/BCc)$. For this we apply the subclaim. We claim that $\mathrm{dcl}^P(BCc) = \bigcup \left\{ a^P_{i,j} \,|\, i < i_0, j < \omega \right\} \cup B^P \cup c$. Why? Choose some basis D for BC over P such that D contains a basis for B over P. If some element x in C is in $\mathrm{acl}(DP)$, then by indiscerniblity, $x \in \mathrm{acl}((a_{i,j} \cap D) \cup BP)$ for some i,j, which means that $x \in \mathrm{acl}(P \cup ((a_{i,j}B) \cap D))$, so the tuple from P that witnesses this is already in $a^P_{i,j}$. Similarly, $\mathrm{dcl}^P(a_{i_0,j}BCc) = a^P_{i_0,j} \cup \mathrm{dcl}^P(BC) \cup c$. By the subclaim above, we are done. □

Remark 2.14. With the same proof, one can show that if T is strongly minimal, and $P = \{a_i \mid i < \omega\}$ is an infinite indiscernible set in $M \models T$ of cardinality $\aleph_1$, the theory of the structure $\langle M, P, < \rangle$ where $<$ is some dense linear order with no end points on P, is strongly^2 dependent.

We finish this section with the following conjecture:

Conjecture 2.15. *All strongly^2 dependent groups are stable, i.e. if G is a group such that $Th(G, \cdot)$ is strongly^2 dependent, then it is stable.*

Example 2.9 and Corollary 2.8 show that this might be reasonable. This is related to the conjecture of Shelah in [Shec] that all strongly^2 dependent infinite fields are algebraically closed.

3. Baldwin-Saxl type lemmas

The next lemma is the type definable version of the Baldwin-Saxl Lemma (see Lemma 1.3). But first,

Notation 3.1. If $p(x, y)$ is a partial type, then $|p|$ is the size of the set of formulas $\varphi(x, z_1, \dots, z_n)$ (where z_i is a singleton) such that for some finite tuple $y_1, \dots, y_n \in y$, $\varphi(x, y_1, \dots, y_n) \in p$. In this sense, the size of any type is bounded by $|T|$.

Lemma 3.2. *Suppose G is a type definable group in a dependent theory T.*

1. *If $p_i(x, y_i)$ is a type of for $i < \kappa$ (y_i may be an infinite tuple), $|\bigcup p_i| < \kappa$, and $\langle c_i \mid i < \kappa \rangle$ is a sequence of tuples such that $p_i(\mathfrak{C}, c_i)$ is a subgroup of G, then for some $i_0 < \kappa$, $\bigcap_{i<\kappa} p_i(\mathfrak{C}, c_i) = \bigcap_{i<\kappa, i \neq i_0} p_i(\mathfrak{C}, c_i)$.*
2. *In particular, Given a family of uniformly type definable subgroups, defined by $p(x, y)$, and C of size $|p|^+$, there is some $c_0 \in C$ such that $\bigcap_{c \neq c_0} p(\mathfrak{C}, c) = \bigcap_{c \in C} p(\mathfrak{C}, c)$.*
3. *In particular, if $\left\{G_i \,\middle|\, i < |T|^+\right\}$ is a family of type definable subgroups (defined with parameters), then there is some $i_0 < |T|^+$ such that $\bigcap G_i = \bigcap_{i \neq i_0} G_i$.*

Proof. (1) Denote $H_i = p_i(\mathfrak{C}, c_i)$. Suppose not, i.e. for all $i < \kappa$, there is some g_i such that $g_i \in H_j$ iff $i \neq j$. If $d_1, d_2 \in H_i$ then $d_1 \cdot g_i \cdot d_2 \notin H_i$. Hence by compactness there is some formula φ_i, $\varphi_i(x, c_i) \in p_i(x, c_i)$ such that for all such $d_1, d_2 \in H_i$, $\neg\varphi_i(d_1 g_i d_2, c_i)$ holds. Since $|\bigcup p_i| < \kappa$, we may assume that for $i < \omega$, φ_i is constant and equals $\varphi(x, y)$. Now for any finite subset $s \subseteq \omega$, let $g_s = \prod_{i \in s} g_i$ (the order does not matter). So we have $\varphi(g_s, c_i)$ iff $i \notin s$ — a contradiction.

(2) and (3) now follow easily from (1). $\square$

In (2) of Lemma 3.2, if C is an indiscernible sequence, then the situation is simpler:

Corollary 3.3. *Suppose* G *is a type definable group in a dependent theory* T*. Given a family of uniformly type definable subgroups, defined by* $p(x,y)$*, and an indiscernible sequence* $C = \langle a_i \mid i \in \mathbb{Z} \rangle$*,* $\bigcap_{i \neq 0} p(\mathfrak{C}, a_i) = \bigcap_{i \in \mathbb{Z}} p(\mathfrak{C}, a_i)$*.*

Proof. Assume not. By indiscernibility, we get that for all $i \in \mathbb{Z}$, $\bigcap_{j \neq i} p(\mathfrak{C}, a_j) = p(\mathfrak{C}, a_i)$. Let I be an indiscernible sequence which extends C to length $|p|^+$. Then by indiscernibility and compactness the same is true for this sequence. This contradicts Lemma 3.2. □

Remark 3.4. In the proof that G^{00} exists in dependent theories, the above corollary is in the kernel of the proof.

If T is strongly dependent, and C is indiscernible, we can even assume that the order type is ω. Let us recall,

Definition 3.5. A theory T is said to be <u>not</u> *strongly dependent* if there exists a sequence of formulas $\langle \varphi_i(x, y_i) \mid i < \omega \rangle$ and an array $\langle a_{i,j} \mid i, j < \omega \rangle$ such that

- The array $\langle a_{i,j} \mid i, j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x, a_{i,0}) \wedge \neg\varphi_i(x, a_{i,1}) \mid i < \omega\}$ is consistent.

So T is *strongly dependent* when this configuration does not exist.

Lemma 3.6. *Suppose* G *is a type definable group in a strongly dependent theory* T*. Given a family of type definable subgroups* $\{p_i(x, a_i) \mid i < \omega\}$ *such that* $\langle a_i \mid i < \omega \rangle$ *is an indiscernible sequence and* $p_{2i} = p_{2i+1}$ *for all* $i < \omega$*, there is some* $i < \omega$ *such that* $\bigcap_{j \neq i} p_j(\mathfrak{C}, a_j) = \bigcap_{j < \omega} p_j(\mathfrak{C}, a_j)$*.*

In particular, this is true when p *is constant.*

Proof. Denote $H_i = p_i(\mathfrak{C}, a_i)$. Assume not, i.e. for all $i < \omega$, there exists some $g_i \in G$ such that $g_i \in H_j$ iff $i \neq j$. For each even $i < \omega$ we find a formula $\varphi_i(x,y) \in p_i(x,y)$ such that for all $d_1, d_2 \in H_i$, $\neg\varphi_i(d_1 g_i d_2, a_i)$. Let $n < \omega$, and consider the product $g_n = \prod_{i<n,\, 2|i} g_i$ (the order does not matter). Then for odd $i < n$, $\varphi_{i-1}(g_n, a_i)$ holds (because $\varphi_{i-1} \in p_{i-1} = p_i$ by assumption), and for even $i < n$, $\neg\varphi_i(g_n, a_i)$ holds. By compactness, we can find $g \in G$ such that $\varphi_{i-1}(g_n, a_i)$ holds for all odd $i < \omega$ and $\neg\varphi_i(g, a_i)$ for all even $i < \omega$. Now expand the sequence by adding a sequence $\langle b_{i,j} \mid j < \omega \rangle$ after each pair a_{2i}, a_{2i+1}. Then the array defined by $a_{i,0} = a_{2i}$, $a_{i,1} = a_{2i+1}$ and $a_{i,j} = b_{i,j-2}$ for $j \geq 2$ will show that the theory is not strongly dependent. □

If the theory is of bounded dp-rank, then we can say even more.

Definition 3.7. A theory T is said to have *bounded dp-rank*, if there is some $n < \omega$ such that the following configuration does <u>not</u> exist: a sequence of formulas $\langle \varphi_i(x, y_i) \mid i < n \rangle$ where x is a <u>singleton</u> and an array $\langle a_{i,j} \mid i < n, j < \omega \rangle$ such that

- The array $\langle a_{i,j} \,|\, i < n, j < \omega \rangle$ is an indiscernible array (over $\emptyset$).
- The set $\{\varphi_i(x, a_{i,0}) \wedge \neg\varphi_i(x, a_{i,1}) \,|\, i < n\}$ is consistent.

T is *dp-minimal* if $n = 2$.

Note that if T has bounded dp-rank, then it is strongly dependent.

Remark 3.8. All dp-minimal theories are of bounded dp-rank. This includes all o-minimal theories and the p-adics.

The name is justified by the following fact:

Fact 3.9. [UOK] *If T has bounded dp-rank, then for any $m < \omega$, there is some $n_m < \omega$ such that a configuration as in Definition 3.7 with n_m replacing n is impossible for a tuple x of length m (in fact $n_m \leq m \cdot n_1$).*

Lemma 3.10. *Let G be type definable group in a bounded dp-rank theory T.*

Given a family of type definable subgroups $\{p_i(x, a_i) \,|\, i < \omega\}$ such that $\langle a_i \,|\, i < \omega \rangle$ is an indiscernible sequence and $p_{2i} = p_{2i+1}$ for all $i < \omega$, there is some $n < \omega$ and $i < n$ such that $\bigcap_{j \neq i, j < n} p_j(\mathfrak{C}, a_j) = \bigcap_{j < n} p_j(\mathfrak{C}, a_j)$.

In particular, if p_i is constant (say p) and $\langle a_i \,|\, i < \omega \rangle$ is an indiscernible set, then $\bigcap_{i<\omega} p(\mathfrak{C}, a_i) = \bigcap_{i<n} p(\mathfrak{C}, a_i)$.

In particular, T has Property A.

Proof. The proof is exactly the same as the proof of Lemma 3.6, but we only need to construct g_n for n large enough. $\square$

Another similar proposition:

Proposition 3.11. *Assume T is strongly dependent, G a type definable group and $G_i \leq G$ are type definable normal subgroups for $i < \omega$. Then there is some i_0 such that $\left[\bigcap_{i \neq i_0} G_i : \bigcap_{i<\omega} G_i\right] < \infty$.*

Proof. Assume not. Then, for each $i < \omega$, we have an indiscernible sequence $\langle a_{i,j} \,|\, j < \omega \rangle$ (over the parameters defining all the groups) such that $a_{i,j} \in \bigcap_{k \neq i} G_k$ and for $j_1 < j_2 < \omega$, $a_{i,j_1}^{-1} \cdot a_{i,j_2} \notin G_i$. Note that if $d_1, d_2, d_3 \in G_i$, then $d_1 \cdot a_{i,j_1}^{-1} \cdot d_2 \cdot a_{i,j_2} \cdot d_3 \notin G_i$, since G_i is normal. By compactness there is a formula $\psi_i(x)$ in the type defining G_i such that for all $d_1, d_2, d_3 \in G_i$, $\neg\psi_i\left(d_1 \cdot a_{i,j_1}^{-1} \cdot d_2 \cdot a_{i,j_2} \cdot d_3\right)$ holds (by indiscernibility it is the same for all $j_1 < j_2$). We may assume, applying Ramsey, that the array $\langle a_{i,j} \,|\, i, j < \omega \rangle$ is indiscernible (i.e. the sequences are mutually indiscernible). Let $\varphi_i(x, y) = \psi_i\left(x^{-1} \cdot y\right)$.

Now we check that the set $\{\varphi_i(x, a_{i,0}) \wedge \neg\varphi_i(x, a_{i,1}) \,|\, i < n\}$ is consistent for each $n < \omega$. Let $c = a_{0,0} \cdot \ldots \cdot a_{n-1,0}$ (the order does not really matter, but for the proof it is easier to fix one). So $\varphi_i(c, a_{i,0})$ holds iff $\psi_i\left(a_{n-1,0}^{-1} \cdot \ldots \cdot a_{i,0}^{-1} \cdot \ldots \cdot a_{0,0}^{-1} \cdot a_{i,0}\right)$ holds. But since G_i is normal,

$a_{i,0}^{-1} \cdot \ldots \cdot a_{0,0}^{-1} \cdot a_{i,0} \in G_i$, so the entire product is in G_i, so $\varphi_i(c, a_{i,0})$ holds. On the other hand, $\psi_i\left(a_{n-1,0}^{-1} \cdot \ldots \cdot a_{i,0}^{-1} \cdot \ldots \cdot a_{0,0}^{-1} \cdot a_{i,1}\right)$ does not hold by choice of ψ_i. □

The following Corollary is a weaker version of Corollary 2.7:

Corollary 3.12. *If G is an abelian definable group in a strongly dependent theory and $S \subseteq \omega$ is an infinite set of pairwise co-prime numbers, then for almost all (i.e. for all but finitely many) $n \in S$, $[G : G^n] < \infty$. In particular, if K is a definable field in a strongly dependent theory, then for almost all primes p, $\left[K^\times : (K^\times)^p\right] < \infty$.*

Proof. Let $K \subseteq S$ be the set of $n \in S$ such that $[G : G^n] < \infty$. If $S \backslash K$ is infinite, we replace S with $S \backslash K$.

For $i \in S$, let $G_i = G^i$ (so it is definable). By Proposition 3.11, there is some n such that $\left[\bigcap_{i \neq n} G_i : \bigcap_{i \in S} G_i\right] < \infty$. If $[G : G_n] = \infty$, then there is an indiscernible sequence $\langle a_i \,|\, i < \omega \rangle$ of elements of G, such that $a_i^{-1} \cdot a_j \notin G_n$. Suppose $S_0 \subseteq S \backslash \{n\}$ is a finite subset and let $r = \prod S_0$. Then $\langle a_i^r \,|\, i < \omega \rangle$ is an indiscernible sequence in $G^r \subseteq \bigcap_{i \in S_0} G_i$ such that $a_i^{-r} \cdot a_j^r \notin G_n$. So by compactness, we can find such a sequence in $\bigcap_{i \neq n} G_i$ — contradiction. □

Remark 3.13. The above Proposition and Corollary can be generalized (with almost the same proofs) to the case where the theory is only *strong*. For the definition, see [Adl].

Remark 3.14. This Corollary generalizes in some sense [KP, Proposition 2.1] (as they only assumed finite weight of the generic type). And so, as in [KP, Corollary 2.2] , we can conclude that if K is a field definable in a strongly stable theory (i.e. the theory is strongly dependent and stable), then $K^p = K$ for almost all primes p.

Problem 3.15. Is Proposition 3.11 is still true without the assumption that the groups are normal?

Note that in strongly dependent[2] theories, this assumption is not needed: Let $H_i = \bigcap_{j<i} G_i$. Then $[H_i : H_{i+1}] < \infty$ for all i big enough by Proposition 2.4. But this implies $\left[\bigcap_{j \neq i} G_j : \bigcap_j G_j\right] < \infty$.

κ-intersection.

This part is joint work with Frank Wagner.

Definition 3.16. For a cardinal κ and a family $\mathfrak{F}$ of subgroups of a group G, the κ intersection $\bigcap_\kappa \mathfrak{F}$ is $\{g \in G \,|\, |\{F \in \mathfrak{F} \,|\, g \notin F\}| < \kappa\}$.

Proposition 3.17. *Let G be a type definable group in a dependent theory. Suppose*

- *$\mathfrak{F}$ is a family of uniformly type definable subgroups defined by $p(x, y)$.*

Then for any regular cardinal $\kappa > |p|$ *(in the sense of Notation 3.1), and any subfamily* $\mathfrak{G} \subseteq \mathfrak{F}$*, there is some* $\mathfrak{G}' \subseteq \mathfrak{G}$ *such that*

$\star$ $|\mathfrak{G}'| < \kappa$ *and* $\bigcap \mathfrak{G}$ *is* $\bigcap \mathfrak{G}' \cap \bigcap_\kappa \mathfrak{G}$*.*

Proof. Let κ be such a cardinal. Assume that there is some family $\mathfrak{G} = \{H_i \,|\, i < \varkappa\}$, which is a counterexample of the proposition. For $g \in G$, let $J_g = \{i < \varkappa \,|\, g \in H_i\}$. So $g \in \bigcap_\kappa \mathfrak{G}$ iff $|\varkappa \backslash J_g| < \kappa$.

For $i < \kappa$ we define by induction $g_i \in \bigcap_\kappa \mathfrak{G}$, $I_i \subseteq \varkappa$, $R_i \subseteq \varkappa$ and $\alpha_i < \varkappa$ such that

(1) $R_0 = [0, \alpha_0)$ and for $0 < i$, $R_i = \bigcup_{j<i} R_j \cup \left[[\sup_{j<i} \alpha_j, \alpha_i) \cap \bigcap_{j<i} I_j\right]$ (so $R_i \subseteq \alpha_i$)

(2) $\bigcap_{j \le i} J_{g_j} \subseteq R_i \cup I_i$ (so by the definition of $\bigcap_\kappa$, and by the regularity of κ,$|\varkappa \backslash (R_i \cup I_i)| < \kappa$)

(3) $\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{\alpha \in R_i} H_\alpha$

(4) $I_i \cap [0, \alpha_i] = \emptyset$

(5) I_i is $\subseteq$-decreasing

(6) α_i is $<$-increasing

(7) $I_i \subseteq J_{g_i}$

(8) For $j < i$, $g_i \in H_{\alpha_j}$, $g_j \in H_{\alpha_i}$ and $g_i \notin H_{\alpha_i}$

Let $\alpha_0 < \varkappa$ be minimal such that there is some $g_0 \in \bigcap_\kappa \mathfrak{G} \backslash H_{\alpha_0}$ (it must exist, otherwise $\bigcap_\kappa \mathfrak{G} = \bigcap \mathfrak{G}$). Let $I_0 = \{j > \alpha_0 \,|\, g_{\alpha_0} \in H_j\}$.

For α_0, (2), (3) and (4) are true, by the definition of $\bigcap_\kappa$ and the choice of α_0.

Suppose we have chosen g_j, I_j and α_j (so R_j is already defined by (1)) for $j < i$.

Let $J = \bigcap_{j<i} I_j$. Choose $g_i \in \left(\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j}\right) \backslash H_{\alpha_i}$ where $\alpha_i \in J$ is the smallest possible such that this set is nonempty. Suppose for contradiction that we cannot find such α_i, then $\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{\alpha \in J} H_\alpha$, so

$$\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \cap \bigcap_{j \in \varkappa \backslash J} H_j = \bigcap \mathfrak{G}.$$

Let $J' = J \cup \bigcup_{j<i} R_j$, then by (3), $\bigcap \mathfrak{G}$ equals

$$\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \cap \bigcap_{j \in \varkappa \backslash J'} H_j.$$

Note that $\bigcap_{j<i} (R_j \cup I_j) \subseteq J'$, so by regularity of κ, and by (2), $|\varkappa \backslash J'| < \kappa$, so we get a contradiction.

Let $I_i = \{\alpha_i < j \in J \,|\, g_i \in H_j\}$, and let us check the conditions above.

Conditions (4) – (7) are easy.

Condition (2): By induction we have

$$\bigcap_{j \le i} J_{g_j} = \bigcap_{j<i} J_{g_j} \cap J_{g_i} \subseteq J' \cap J_{g_i} \subseteq R_i \cup (J \cap J_{g_i})$$

But by (4) and the definition of R_i, letting $\alpha = \sup_{j<i} \alpha_j$, we have

$$J \cap J_{g_i} \subseteq \left[[\alpha, \alpha_i) \cap \bigcap_{j<i} I_j\right] \cup I_i \subseteq R_i \cup I_i$$

Condition (3) is true by the minimality of α_i: $\bigcap_\kappa \mathfrak{G} \cap \bigcap_{j<i} H_{\alpha_j} \subseteq \bigcap_{\beta \in J \cap [\alpha, \alpha_i)} H_\beta$, so by the induction hypothesis, we are done.

Condition (8): We show that $g_j \in H_{\alpha_i}$ for $j < i$. We have that $\alpha_i \in J$ so also in I_j which, by (7) is a subset of J_{g_j}, so $g_j \in H_{\alpha_i}$.

Finally, we have that for each $i, j < \kappa$, $g_i \in H_{\alpha_j}$ iff $i \neq j$. But by Lemma 3.2, there is some $i_0 < |p|^+$ such that $\bigcap_{i \neq i_0} H_{\alpha_i} = \bigcap_{i < |p|^+} H_{\alpha_i}$ — contradiction. □

4. A counterexample

In this section we shall present an example that shows that Property A does not hold in general dependent (or even stable) theories.

Let $S = \{u \subseteq \omega \,|\, |u| < \omega\}$, and $V = \{f : S \to 2 \,|\, |\mathrm{supp}(f)| < \infty\}$ where $\mathrm{supp}(f) = \{x \in S \,|\, f(x) \neq 0\}$. This has a natural group structure as a vector space over $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$.

For $n, m < \omega$, define the following groups:

- $G_n = \{f \in V \,|\, u \in \mathrm{supp}(f) \Rightarrow |u| = n\}$
- $G_\omega = \prod_n G_n$
- $G_{n,m} = \{f \in V \,|\, u \in \mathrm{supp}(f) \Rightarrow |u| = n \,\&\, m \in u\}$ (so $G_{0,m} = 0$)
- $H_{n,m} = \{\eta \in G_\omega \,|\, \eta(n) \in G_{n,m}\}$

Now we construct the model:

Let L be the language (vocabulary) $\{P, Q\} \cup \{R_n \,|\, n < \omega\} \cup L_{AG}$ where L_{AG} is the language of abelian groups, $\{0, +\}$; P and Q are unary predicates; and R_n is binary. Let M be the following L-structure: $P^M = G_\omega$ (with the group structure), $Q^M = \omega$ and $R_n = \{(\eta, m) \,|\, \eta \in H_{n,m}\}$. Let $T = Th(M)$.

Let $p(x, y)$ be the type $\bigcup\{R_n(x, y) \,|\, n < \omega\}$. Note that since $H_{n,m}$ is a subgroup of G_ω, for each $m < \omega$, $p(M, m)$ is a subgroup of G_ω.

Claim 4.1. Let $N \models T$ be $\aleph_1$-saturated. For any m, and any distinct $\alpha_0, \dots, \alpha_m \in P^N$, $\bigcap_{i \leq m} p(N, \alpha_i)$ is different than any sub-intersection of size m.

Proof. We show that $\bigcap_{i \leq m} p(N, \alpha_i) \subsetneq \bigcap_{i<m} p(N, \alpha_i)$ (the general case is similar). More specifically, we show that

$$\bigcap_{i<m} p(N, \alpha_i) \setminus \bigcap_{i \leq m} R_m(N, \alpha_i) \neq \emptyset.$$

By saturation, it is enough to show that this is the case in M, so we assume $M = N$. Note that if $\eta \in \bigcap_{i \leq m} R_m(M, \alpha_i)$, then $\eta \in H_{m,\alpha_i}$ for all $i \leq m$. So for all $i \leq m$, $u \in \mathrm{supp}(\eta(n)) \Rightarrow |u| = m \,\&\, \alpha_i \in u$. This implies that $\mathrm{supp}(\eta(m)) = \emptyset$, i.e. $\eta(m) = 0$. But we can find $\eta \in \bigcap_{i<m} p(M, \alpha_i)$ such that $\eta(m) \neq 0$, for instance let $\eta(n) = 0$ for all $n \neq m$ while $|\mathrm{supp}(\eta(m))| = 1$ and $\eta(m)(\{\alpha_0, \dots, \alpha_{m-1}\}) = 1$. □

Next we shall show that T is stable. For this we will use κ resplendent models. This is a very useful (though not a very well known) tool for proving that theories are stable, and we take the opportunity to promote it.

Definition 4.2. Let κ be a cardinal. A model M is called κ-resplendent if whenever

- $M \prec N$; N' is an expansion of N by less than κ many symbols; $\bar{c}$ is a tuple of elements from M and $\lg(\bar{c}) < \kappa$

There exists an expansion M' of M to the language of N' such that $\langle M', \bar{c}\rangle \equiv \langle N', \bar{c}\rangle$.

The following remarks are not crucial for the rest of the proof.

Remark 4.3. [Sheb]

(1) If κ is regular and $\kappa > |T|$, and $\lambda = \lambda^{<\kappa}$, then T has a κ-resplendent model of size λ.
(2) A κ resplendent model is also κ-saturated.
(3) If M is κ resplendent then M^{eq} is also such.

The following is a useful observation:

Claim 4.4. If M is κ-resplendent for some κ, and $A \subseteq M$ is definable and infinite, then $|A| = |M|$.

Proof. Enrich the language with a function symbol f. Let $T' = T \cup \{f : M \to A \text{ is injective}\}$. Then T' is consistent with an elementary extension of M (for example, take an extension N of M where $|A| = |M|$, and then take an elementary substructure $N' \prec N$ of size $|M|$ containing M and A^N). Hence we can expand M to a model of T'. □

The main fact is

Theorem 4.5. [Sheb, Main Lemma 1.9]*Assume κ is regular and $\lambda = \lambda^{\kappa} + 2^{|T|}$. Then, if T is unstable then T has $> \lambda$ pairwise nonisomorphic κ-resplendent models of size λ*[1]*. On the other hand, if T is stable and $\kappa \geq \kappa(T) + \aleph_1$ then every κ-resplendent model is saturated.*

Proposition 4.6. T *is stable.*

Proof. We may restrict T to a finite sub-language, $L_n = \{P, Q,\} \cup \{R_i \,|\, i < n\} \cup L_{AG}$.

Our strategy is to prove that our theory has a unique model in size λ which is κ resplendent where $\kappa = \aleph_0$, $\lambda = 2^{\aleph_0}$. Let N_0, N_1 be two κ-resplendent models of size λ.

By Claim 4.4, $\left|Q^{N_0}\right| = \left|Q^{N_1}\right| = \lambda$ and we may assume that $Q^{N_0} = Q^{N_1} = \lambda$.

Let $G_0 = P^{N_0}$ and $G_1 = P^{N_1}$ with the group structure. For $i < n$, $j < 2$ and $\alpha < \lambda$, let $H^j_{i,\alpha} = \left\{x \in G_j \,\middle|\, R_i^{N_j}(x, \alpha)\right\}$. This is a definable subgroup of G_j. For $k \leq n$, let $G^k_j = \bigcap_{\alpha<\lambda,\, i\neq k,\, i<n} H^j_{i,\alpha}$. In our original model M, this group is $\{\eta \in G_\omega \,|\, \forall i \neq k, i < n\, (\eta(i) = 0)\}$.

[1]In fact, by [Sheb, Claim 3.1], if T is unstable there are 2^λ such models.

Note that $G_j = \sum_{k<n} G_j^k$, and that $G_j^{k_0} \cap \sum_{k<n, k\neq k_0} G_j^k = G_j^n$ (this is true in our original model M, so it is part of the theory). We give each G_j^k the induced L-structure $N_j^k = \left\langle G_j^k, \lambda \right\rangle$, i.e. we interpret $R_i^{N_j^k} = R_i \cap \left(G_k^j \times \lambda\right)$.

Since these groups are definable and infinite, their cardinality is λ, and hence their dimension (over $\mathbb{F}_2$) is λ. In particular there is a group isomorphism $f_n : G_0^n \to G_1^n$. Note that f_n is an isomorphism of the induced structure on $N_j^n = \left\langle G_j^n, \lambda \right\rangle$.

Claim. For $k < n$, there is an isomorphism $f_k : G_0^k \to G_1^k$ which is an isomorphism of the induced structure $N_j^k = \left\langle G_j^k, \lambda \right\rangle$ and extends f_n.

Assuming this claim, we shall finish the proof. Define $f : G_0 \to G_1$ by: given $x \in G_0$, write it as a sum $\sum_{k<n} x_k$ where $x_k \in G_0^k$, and define $f(x) = \sum_{k<n} f(x_k)$. This is well defined because if $\sum_{k<n} x_k = \sum_{k<n} x'_k$ then $\sum_{k<n} (x_k - x'_k) = 0$ so for all $k < n$, $x_k - x'_k \in G_0^n$, so

$$\begin{aligned} \sum_{k<n} \left(f(x_k) - f(x'_k)\right) &= \sum_{k<n} \left(f(x_k - x'_k)\right) = \sum_{k<n} \left(f_n(x_k - x'_k)\right) = \\ &= f_n\left(\sum_{k<n} x_k - x'_k\right) = f_n(0) = 0. \end{aligned}$$

It is easy to check similarly that f is a group isomorphism. Also, f is an L_n-isomorphism because if $R_i^{N_0}(a, \alpha)$ for some $i < n$, $\alpha < \lambda$ and $a \in G_0$, then write $a = \sum_{k<n} a_k$ where $a_k \in G_0^k$. Since $R_i^{N_0}(a, \alpha)$ and $R_i^{N_0}(a_k, \alpha)$ for all $k \neq i$, it follows that $R_i^{N_0}(a_i, \alpha)$ holds, so $R_i^{N_1}(f_k(a_k), \alpha)$ holds for all $k < n$, and so $R_i^{N_1}(f(a), \alpha)$ holds. The other direction is similar.

Proof. (of claim) For a finite set b of elements of λ, let $L_b^j = G_j^k \cap \bigcap_{\alpha\in b} H_{k,\alpha}^j$. For $m \leq k+1$, let $K_m^j = \sum_{|b|=m} L_b^j$ (as a subspace of G_k^j), so K_m^j is not necessarily definable (however K_0^j and K_{k+1}^j are). So this is a decreasing sequence of subgroups (so subspaces), $G_j^k = K_0^j \geq \ldots \geq K_{k+1}^j = G_j^n$. Now it is enough to show that

Subclaim. For $m \leq k+1$, there is an isomorphism $f_m : K_m^0 \to K_m^1$ which is an isomorphism of the induced structure $\left\langle K_m^j, \lambda \right\rangle$.

Proof. (of subclaim) The proof is by reverse induction. For $m = k+1$ we already have this. Suppose we have f_{m+1} and we want to construct f_m. Let $b \subseteq \lambda$ of size m. If $m = k$, then it is easy to see that $\left|L_b^j / \left(K_{m+1}^j \cap L_b^j\right)\right| = 2$ (this is true in M), so there is an isomorphism $g_b : L_b^0 / \left(K_{m+1}^0 \cap L_b^0\right) \to L_b^1 / \left(K_{m+1}^1 \cap L_b^1\right)$.

Assume $|b| < k$. In our original model M, $L_b \subseteq K_k$, but here can find infinitely pairwise distinct cosets in $L_b^j / \left(K_{m+1}^j \cap L_b^j\right)$. Indeed, we can write a type in λ infinitely many variables $\{x_i \mid i < \lambda\}$ over b saying that $x_i \in L_b$ and $x_i - x_j \notin K_{m+1}$ for $i \neq j$ — for all $r < \omega$, it will contain a formula

of the form

$$\forall\left(z_0,\ldots,z_{r-1}\right)\forall_{t<r}\left(\bar{y}_t\right)\left(\left[\forall t<r\left(z_t\in L_{\bar{y}_t}\wedge\left|\bar{y}_t\right|=m+1\right)\right]\to x_i-x_j\neq\sum_{t=0}^{r-1}z_t\right).$$

To show that this type is consistent, we may assume that $b\subseteq Q^M$ so we work in our original model M. For such r and b, choose distinct $\eta_0,\ldots\eta_{l-1}\in G_\omega$ such that for $s,s'<l$

- $\eta_s(i)=0$ for $i\neq k$
- $\left|\operatorname{supp}\left(\eta_s(k)\right)\right|=r+1$
- $u_1\in\operatorname{supp}\left(\eta_s(k)\right)\,\&\,u_2\in\operatorname{supp}\left(\eta_{s'}(k)\right)\Rightarrow u_1\cap u_2=b$ (s might be equal to s')

Then $\{\eta_s\,|\,s<l\}$ is such that η_{s_1},η_{s_2} satisfies the formula above for all $s_1\neq s_2<l$ (assume $z_0\in L_{c_0},\ldots,z_{r-1}\in L_{c_r}$ where $|c_t|=m+1$ and $\sum_{t<r}z_t=\eta_{s_1}-\eta_{s_2}$. We may assume that

$$\bigcup_{t<r}\operatorname{supp}\left(z_t\right)=\operatorname{supp}\left(\eta_{s_1}-\eta_{s_2}\right)=\operatorname{supp}\left(\eta_{s_1}\right)\cup\operatorname{supp}\left(\eta_{s_2}\right),$$

but then for $t<r$, $\left|\operatorname{supp}\left(z_t\right)\right|\leq 1$ by our choice of η_s and this is a contradiction).

Now, let N_j' be an elementary extension of N_j with realizations $D=\{c_i\,|\,i<\lambda\}$ of this type, and we may assume $\left|N_j'\right|=\lambda$. Then, add a predicate for the set D, and an injective function from N_j' to D. Finally, by resplendence of N_j, $\left|L_b^j/\left(K_{m+1}^j\cap L_b^j\right)\right|=\lambda$.

Hence it has a basis of size λ, and let $g_b:L_b^0/\left(K_{m+1}^0\cap L_b^0\right)\to L_b^1/\left(K_{m+1}^1\cap L_b^1\right)$ be an isomorphism of $\mathbb{F}_2$-vector spaces.

Note that $f_{m+1}\restriction K_{m+1}^0\cap L_b^0$ is onto $K_{m+1}^1\cap L_b^1$ (this is because f_{m+1} is an isomorphism of the induced structure). We can write $L_b^j=\left(K_{m+1}^j\cap L_b^j\right)\oplus W^j$ where $W^j\cong L_b^j/\left(K_{m+1}^j\cap L_b^j\right)$, so g_b induces an isomorphism from W^0 to W^1. Now extend $f_{m+1}\restriction K_{m+1}^0\cap L_b^0$ to $f_m^b:L_b^0\to L_b^1$ using g_b.

Next, note that $\left\{L_b^j\,|\,b\subseteq\lambda,\,|b|=m\right\}$ is independent over K_{m+1}^j, i.e. for distinct $b_0,\ldots,b_r$, $L_{b_r}^j\cap\sum_{t<r}L_{b_r}^j\subseteq K_{m+1}^j$. Indeed, in our original model M, the intersection $L_{b_r}\cap\sum_{t<r}L_{b_t}$ is equal to $\sum_{t<r}L_{b_r\cup b_t}$, so this is true also in N_j (in fact, this is true for every choice of finite sets b_t — regardless of their size).

Define f_m as follows: given $a\in K_m^j$, we can write $a=\sum_{b\in B}a_b$ where $a_b\in L_b$ for a finite $B\subseteq\{b\subseteq\lambda\,|\,|b|=m\}$, and define $f_m(a)=\sum f_b\left(a_b\right)$. It is well defined: if $\sum_{b\in B}x_b=\sum_{b'\in B'}y_{b'}$, then for $b_1\in B\cap B'$, $b_2\in B\backslash B'$ and $b_3\in B'\backslash B$, $\left(x_{b_1}-y_{b_1}\right),x_{b_2},y_{b_3}\in K_{m+1}$, so

$$\begin{aligned}&\textstyle\sum_{b\in B}f_b\left(x_b\right)-\sum_{b'\in B'}f_{b'}\left(y_{b'}\right)=\\&\textstyle\sum_{b\in B\cap B'}f_{m+1}\left(x_b-y_b\right)+\sum_{b\in B\backslash B'}f_{m+1}\left(x_b\right)-\sum_{b\in B'\backslash B}f_{m+1}\left(y_b\right)\quad=0.\end{aligned}$$

It is easy to check similarly that f_m is a group isomorphism.

We check that f_m is an isomorphism of the induced structure. So suppose $a\in K_m^0$, $\alpha<\lambda$ and $i<\omega$. If $i\neq k$, then since $K_m^j\subseteq G_j^k$ for $j<2$, both $R_i^{N_0}(a,\alpha)$ and $R_i^{N_1}(f(a),\alpha)$ hold. Suppose $R_k^{N_0}(a,\alpha)$ holds. Write $a=\sum_{b\in B}a_b$ as above. Then (by the remark in parenthesis above) we

may assume that $b \in B \Rightarrow \alpha \in b$. So by definition of f_m, $R_k^{N_1}(f_m(a_\alpha), \alpha)$ holds. The other direction holds similarly and we are done.

$\square$

Note 4.7. This example is not strongly dependent, because the sequence of formulas $R_n(x, y)$ is a witness of that the theory is not strongly dependent. So as we said in the introduction, it is still open whether Property A holds for strongly dependent theories.

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